Mar 15

Last time $|\nabla^k Rm| \leq Ct$

Ricci flow has long time existence

$$g(t) \xrightarrow{C^\infty} g_\infty, \quad (M, g_\infty) \cong (\mathbb{R}^n, \delta_{ij})$$

Thm 1.2 $m(g_\infty) \geq 0$

We define $C^k_\beta := \{ u \in C^k \mid \|u\|_{C^k_\beta} := \sum_{i=0}^{k} \sup r^{-\beta+i} |\nabla^i u| < \infty \}$.

Thm $\sigma' \in (\sigma_0, \boxed{\frac{n-2}{2}})$. $g(t) \xrightarrow{C^k_{-\sigma'}} g_\infty$.

In particular, $(M, g(\infty))$ is asymptotically Euclidean, with the same asy Euclidean chart.

Lemma $\forall k$, $\|\nabla^k Rm\| \leq C_k \cdot t^{-1-\eta_k} r^{-k-\sigma'}$ for some C_k, η_k.

To show this, we define

$$D_k = \{ (x,t) \in M \times [0, \infty) \mid r(x) \geq t^{a_k} \}. \quad a_k > \frac{1}{2}, \text{ to determine later}$$

$\sigma' < \sigma_1 < \sigma_0 < \infty$ implies $(x,t) \notin D_k$

$$|\nabla^k Rm| < C_k t^{1 - \frac{\sigma_0}{2} - \frac{k}{2}} \leq C_k t^{-1-\eta_k} r^{-k-\sigma'}$$

Claim $|\nabla^k Rm| \leq C r^{-4-2\sigma_1-2k}$ $\forall (x,t) \in D_k$. □

Fix $\eta \in C^\infty$, s.t. $\eta \equiv 0$ outside AE end.

$\eta \equiv 1$ if r large

$$m(g(t)) = \lim_{t\to\infty} \int_{S^{n-1}} \eta \underbrace{X(t)}_{(\partial_i g_{ij} - \partial_j g_{ii})\partial_j} \lrcorner \, dV_{g_E}$$

$$= \int \left(\eta \cdot \operatorname{div}(X) + \langle X(t), \nabla\eta \rangle \right) dV. \qquad (\star)$$

$$R = \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + E(g)$$

$E(g)$: polynomial in $g, \partial g$

$E \sim r^{-2\sigma'-2}\partial^2 g$

We have $\| E(g_t) - E(g_\infty) \|(x) \le C \| g(\infty) - g(t) \|_{C^2_{-\sigma}} r_{(x)}^{-2\sigma'-2}$.

At g_∞, $R(\infty) = 0$ $\qquad R(t) = R(t) - R(\infty)$

$$= \operatorname{div} X(t) - \operatorname{div} X(\infty) + E(g(t)) - E(g(\infty)).$$

The above inequality gives

$$|\operatorname{div} X(t) - \operatorname{div} X(\infty) - R(t)| \le C\|g(t) - g(\infty)\|_{C^2_{-\sigma'}}. \qquad (\star\star)$$

Then $m(g(0)) \ge \lim_{t\to\infty} m(g(t))$

$$= \lim_{t\to\infty} m(g(t)) - m(g(\infty))$$

Euclidean 0 mass.

by $(\star)$ and $(\star\star)$

$$\ge \lim_{t\to\infty} \int \eta R(t) - C\eta \cdot \|g(t) - g(\infty)\|_{C^2_{-\sigma}} r^{-2\sigma'-2}$$

$$+ \langle X(t) - X(\infty), \nabla\eta \rangle \, dV$$

$\sigma' \in \left(\frac{n-2}{2}, \sigma\right) \quad 2\sigma' + 2 > n \Rightarrow \eta r^{-2\sigma-2}$ integrable

$$RHS = \lim_{t\to\infty} \int \eta R(t) \geq 0$$

$$\frac{d}{dt}\int R \, dV = \int \Delta R + 2|Ric|^2 - R^2 \, dV.$$

$$\geq -\frac{n-2}{n}\int R^2 \, dV \quad \text{as } |Ric|^2 \geq \frac{R^2}{n}.$$

$$\geq -\frac{C}{(1+t)^{1+\delta}}\int R \, dV$$

since $\lim_{t\to\infty}\int R \, dV = 0 \Leftrightarrow R(t) \equiv 0 \Leftrightarrow M \cong \mathbb{R}^n.$

we are done.

If Ricci flow has long time existence. Thm 1.2 implies $m(g_0) \geq 0$

- T finite time.

$$\Omega := \{ x \in M \mid \limsup_{t\to T} R(x,t) < \infty \} \underset{\text{open}}{\subseteq} M.$$

$$g(t) \xrightarrow{K \subset\subset \Omega} g(T). \quad (\Omega, g(T))$$

- There is $r > 0$. $\forall x \in \Omega \times \{T\}$ with $R(x,T) \geq r^{-2}$.

$$\hat{M} = (M \times [0,T]) \bigcup_{\Omega \times [0,T]} (\Omega \times [0,T]).$$

- $R(T)$ is proper from $\Omega \longrightarrow (0,\infty)$.

Lemma 6.3 $\exists K \subset\subset \Omega$ s.t. $g(T)$ has an AE coordinate on K^c.

Surgery process

$0 < \rho < r'$, let $\Omega_\rho := \{x \in \Omega \mid R(x, T) \le \rho^{-2}\}$

Canonical neighbourhood thm.
Assume M is orientable, if Ω_1 is a ~~compact~~ component of Ω which contains no points of Ω_ρ, we have the following possibilities.

(i) Ω_1 is strong ~~(ε-...)~~ double ε-horn $\quad \Omega_1 \cong S^2 \times \mathbb{R}$

(ii) Ω_1 is a C-capped ε-horn $\quad \Omega_1 \cong \mathbb{R}^3$ or $\mathbb{R}P^2$

(iii) Ω_1 is compact $\quad \Omega_1 \cong S^3/\Gamma$, $S^1 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

$\Omega^0(\rho) = \bigcup$ component of Ω containing points in Ω_ρ.
$= \bigcup$ of AE end and finitely many strong ε-horns.

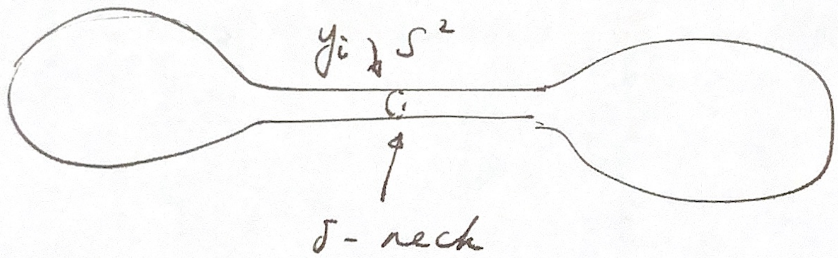
Lemma 6.5 $\delta > 0$, $\exists h \in (0, \delta\rho)$, $D = D(\delta, \rho)$ s.t.
If $x, y, z \in \Omega$ s.t. $R(x, t) \le \rho^{-2}$, $R(y, t) = h^{-2}$, $R(z, t) \ge D h^{-2}$, ~~then~~ $\exists$ a curve connecting x, y, z, then (y, t) is the center of a strong δ-neck.

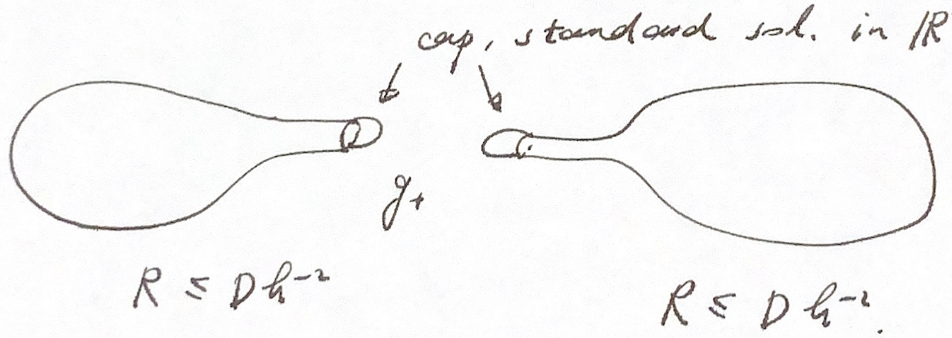
Take $\rho = r'\delta$, (r', δ)-surgery at time T.

By <u>Zorn's lemma</u>, we have maximal collection of $\{N_i\}$ of pairwise δ-neck centered at y_i, with

$R(y_i, T) = h^{-2}$.

Every component of $\Omega \setminus \{N_i\}$ has scalar curvature R either $\le D h^{-2}$ or $\ge \rho^{-2}$. (by Thm 6.5).





(M_+, g_+) is pinched towards positive curve (RF).

For component M_+^0 contains AE, with order σ.

$$m(M_+^0) = m(M).$$

$\longrightarrow$ existence of RF with surgery

Existence of RF with surgery

$\exists$ (M, g_M) on $[0, \infty)$ with (M, g_0) initial manifold

$\delta(t)$ $r(t)$ $k(t)$: $[0, \infty) \to \mathbb{R}^+$. decreasing functions

(i) (M, g_M) has curve pinched toward positive

(ii) the flow satisfies (C, ε) - canonical neighbourhood

(iii) $k(t)$ - noncollapsed on $[0, \infty)$ on scales $\leq \varepsilon$ then

(iv) $\forall t$, the surgery is performed with control $\delta(t)$ at scale $h(t) = h(\rho(t), \delta(t))$.